Exploring Misconceptions with Rich Tasks

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Abstract: *In the following paper, the author investigates Points on Line Segment, a rich task available on the Stella's Stunners section of the* Ohio Resource Center *website. The task was posed to 11 high school students in the greater Columbus area. The author advocates increased use of rich tasks in the secondary school curriculum, discussing ratio and fraction misconceptions that upper level students encountered as they solved the problem.*

Keywords. *problem-solving, ratio, proportional thinking*

1 Introduction

In Fall 2015, I posed a rich mathematical task to high school students as part of an assignment analyzing student thinking and problem solving in an undergraduate methods course. After perusing various internet resources, I came across the *Points on Line Segment* task from Stella's Stunners, a library of rich tasks on the Ohio Resource Center (ORC) website. The task reads as follows.

> Points P and Q are on line segment AB, and both points are on the same side of the midpoint M of AB. Point P divides AB in the ratio $2:3$, and Q divides AB in the ratio $3:4$. If $PQ = 2$, then what is the length of AB? Show all work (ORC, 2016).

Before reading further, I encourage you to solve the task for yourself. You will find the remainder of this paper more meaningful once you have a strong sense of the mathematical nuances of the task.

The *Points on Line Segment* task was presented to students in 8th grade through 12th grade from different schools throughout the Columbus area. At the time of the study, the students were enrolled in math courses ranging from geometry to second-year calculus.

2 Task Content Analysis

2.1 Alignment to Standards

Students considered ratios, distance, fractions, and linear equations as they solve the *Points on the Line Segment* task. As such, the problem addresses the following Common Core State Standards for Mathematics (CCSSO, 2010).

- CCSS.MATH.CONTENT.HSA.CED.A.1: Create equations and inequalities in one variable and use them to solve problems.
- CCSS.MATH.CONTENT.HSA.REI.B.3: Solve linear equations and inequalities in one variable, including equations with coefficients represented by letters.
- CCSS.MATH.CONTENT.HSG.GPE.B.6. Find the point on a directed line segment between two given points that partitions the segment in a given ratio.

The task is not simply a geometry or algebra exercise. Rather, students must combine knowledge from a variety of previous math courses to solve the problem, including setting up and solving algebraic equations, finding common denominators, adding fractions, constructing geometric figures, and interpreting ratios. More specifically, they must know the difference between part-to-whole and part-to-part ratios and how to represent them. Most of this knowledge comes from courses that precede high school geometry. The *Points on Line Segment* task engages students through its familiarity and its challenge. Although the problem may appear rather easy at first glance, students (and many teachers) soon discover that deep knowledge of content and significant interpretation skills are required to solve the task. Although drawing a labeled picture of a line segment is a relatively straightforward exercise, marking ratios - much less knowing what to do with a correctly formatted sketch - is an altogether different challenge. The task is also intriguing because of its "reversed" format. While the standard HSG.GPE.B.6. suggests problems that require students to find points from given ratios, the *Points on Line Segment* task reverses this - giving students a ratio and points while asking them to find information about the whole line segment.

2.2 Misinterpreting Ratios

Recall that the problem states "Point P divides AB in the ratio $2:3$, and Q divides AB in the ratio $3:4$." I suspected that some students might interpret the ratio $2:3$ as the fractional distance of P from A to B (left) rather than the ratio of distances $AP:PB$. A correct interpretation is shown in Figure 1 (right).

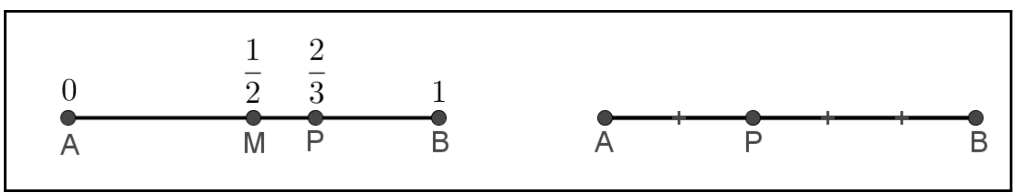


Fig. 1: *(Left) Possible student misconception; (Right) Correct interpretation.*

Given this possible misconception, students in higher-level courses (such as calculus) and geometry were predicted to more likely to answer the task correctly. Calculus students have extensive content and problem solving knowledge. Similarly, geometry students draw figures of various geometric situations routinely. Since the problem is closely linked to CCSSM standard GPE.B.6, one would expect geometry and calculus students to interpret the ratios $2:3$ and $3:4$ correctly.

3 Collected Student Work

3.1 Summary Results

Table 1 provides a summary of the performances of 11 students on the task.

Table 1: *Student performance on* Points on Line Segment *task.*

Student	Grade Level	Current Math Class	Answer
1	12	n/a	70
2	10	Pre-calculus	70
3	12	Calculus	70
4	12	Discrete Math	70
5	12	Calculus	24
6	12	Calculus II	24
7	8	Geometry	24
8	11	Pre-calculus	24
9	8	Geometry	-
10	11	Algebra II	-
11	12	Pre-calculus	-

Figures 2-7 illustrate student work along with my feedback.

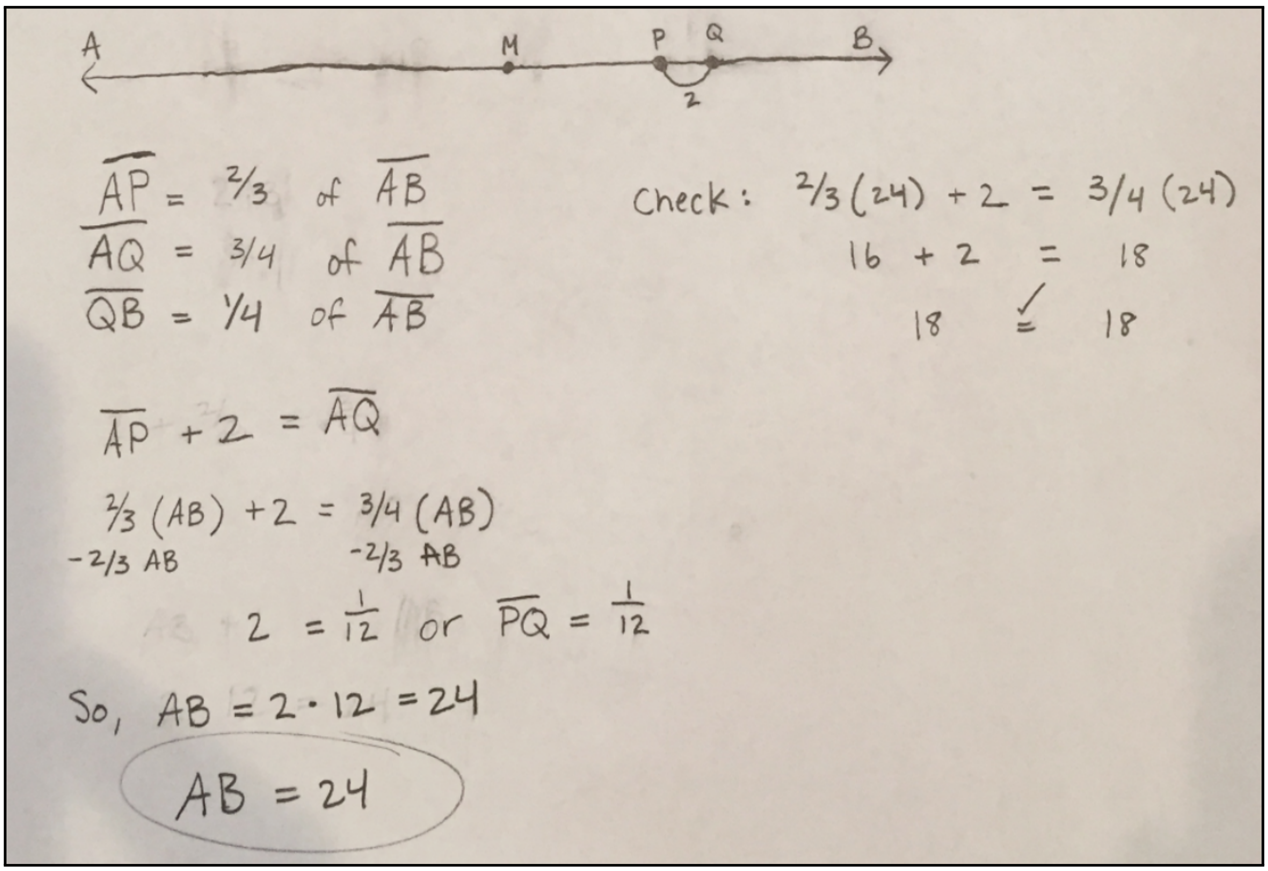


Fig. 2: *Work sample from pre-calculus student.*

Anna Today at 9:30 PM

To:

Re: Another Math Problem

The question states that the point P divides the segment AB in the ratio 2:3. Meaning AP:PB : : 2:3. You did a great job setting up the equation, but adjust your ratios. You might have just misinterpreted the problem (as I did at first too)! Please let me know if you have any questions!

Thanks,
Glover

Fig. 3: *Feedback given to pre-calculus student.*

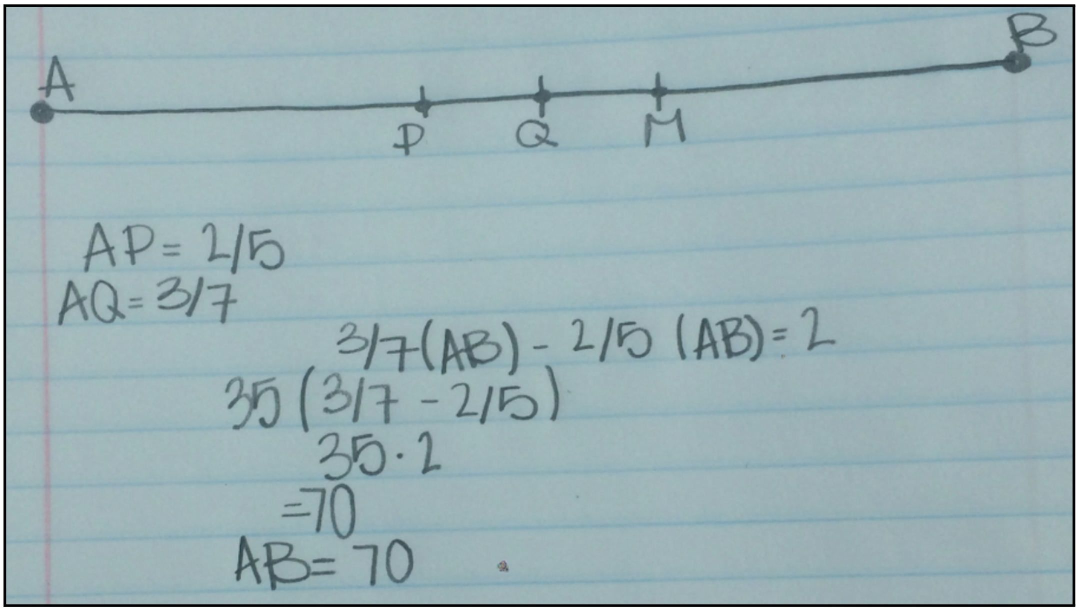


Fig. 4: *Work sample from discrete math student.*

Anna Today at 9:27 PM

To:

Re:

You got the answer correct! But I am having some trouble following your work. It might be that you did a lot of the work in your head. Your thought process seems to be correct, though. Also, make sure you bring down your equal signs on each line! Good job!

Thank you,
Glover

Fig. 5: *Feedback given to discrete math student.*

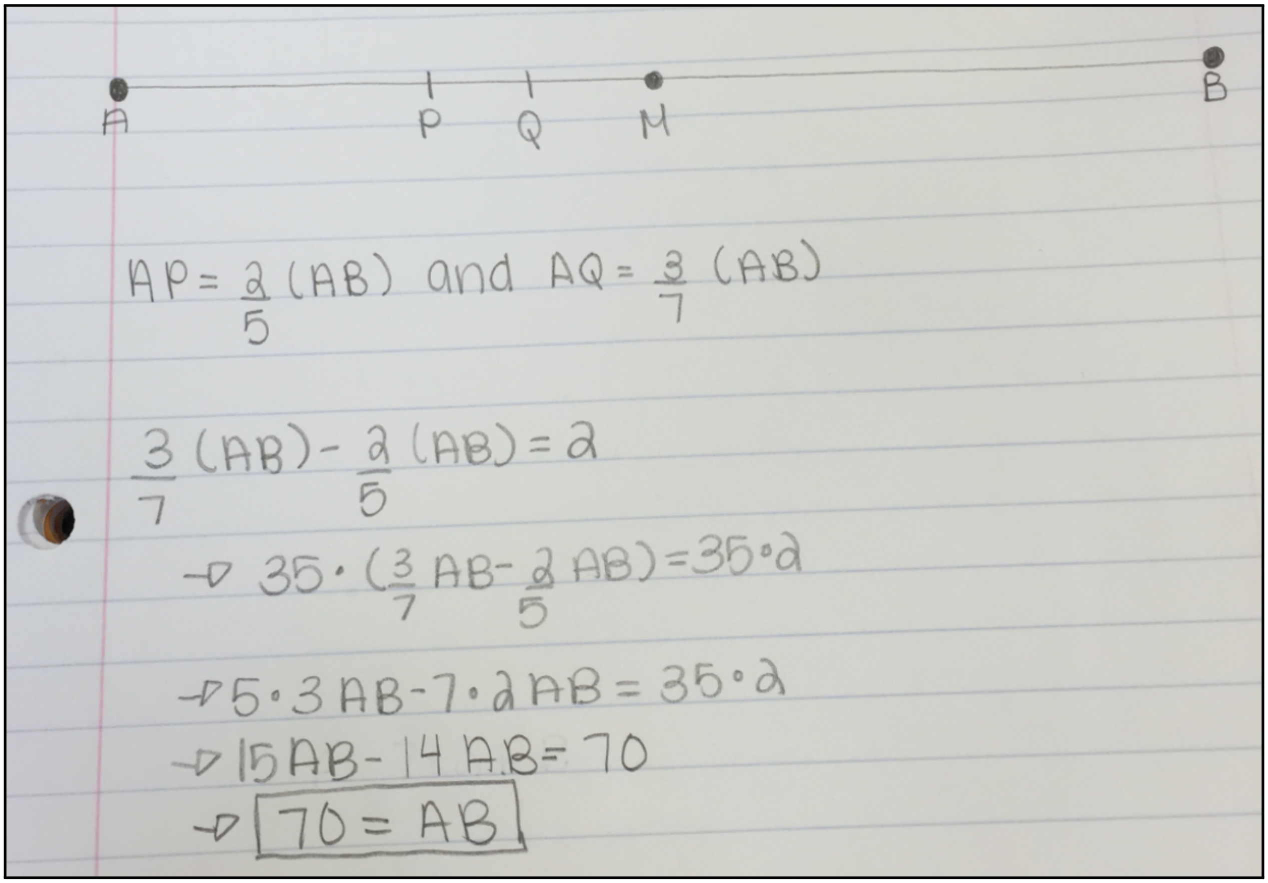
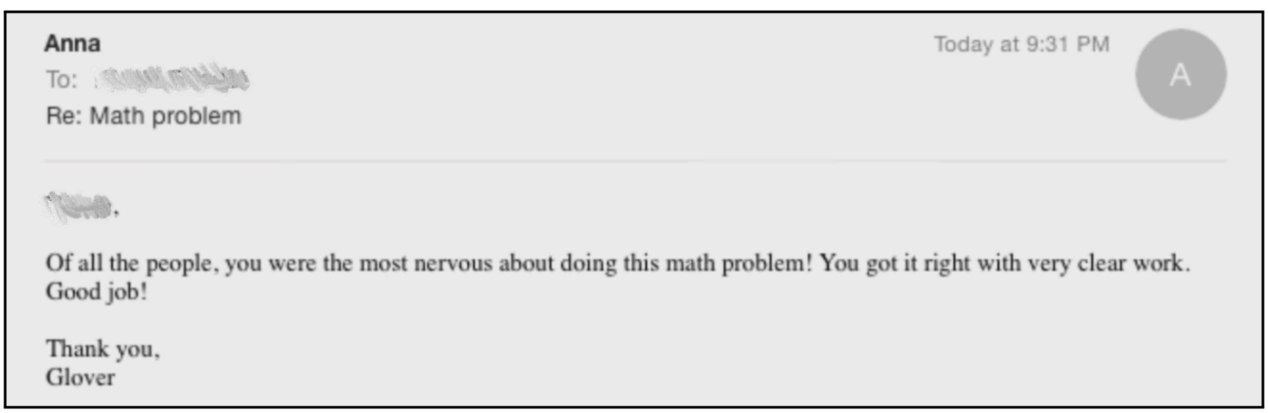


Fig. 6: *Work sample from student not currently enrolled in a mathematics course.*


Anna
Today at 9:31 PM
To: [redacted]
Re: Math problem

[redacted],

Of all the people, you were the most nervous about doing this math problem! You got it right with very clear work. Good job!

Thank you,
Glover

Fig. 7: *Feedback given to student not currently enrolled in a mathematics course.*

4 Interpretations and Observations

The results of the student work were somewhat surprising. Some students were anticipated to make errors with the ratios, but many more did than what was expected. Although our sample size was too small to make generalizations, in this example, the advanced-level math students tended to answer the problem incorrectly, if they attempted the problem at all, while lower-level math students typically experienced more success. The more hesitant the student was, the better he or she performed. Here are actual quotes from two students, both who solved the problem correctly. Although these occurrences might be unique to this problem, rich tasks could be used in any grade level in order to uncover misconceptions in many different topics.

'I didn't learn anything for the good 5 years I was homeschooled, but I'll do my best."

'I didn't have to take math this year since I finished my credits so I might be a little dusty. Don't judge my stupidity."

All students were able to draw a picture and label all points relatively easily. None of the students, however, labeled the segments with the given ratios. It was clear that students understood they had to use fractions of the line segment and somehow relate them to segment $PQ = 2$. Difficulties ensued when students had to interpret what the ratio implied. Note that interpretation of the key word "divides" was central to understanding the task. "Divides" indicates that the point at hand (either P or Q) splits the segment into two pieces such that the segments match the given ratio. Ratios are often represented as fractions. Students in upper level courses often think of ratios and fractions interchangeably. Not surprisingly, a number of participants immediately associated the word "ratio" with a fraction. Students struggled to distinguish when to use part-to-part or part-to-whole ratios. Remarkably, each student who answered the problem incorrectly arrived at the same answer and made the same mistake. For instance, the student whose work is illustrated in Figure 2 interpreted the ratios as part-to-whole, arriving at the answer of 24. The four students who gave 24 as a final answer had virtually identical work, despite the fact that they attended different schools and did not interact with each other in any way. The student whose work is illustrated in Figure 4 interpreted the ratios correctly and answered the problem correctly; however, the student's train of thought was difficult to follow. Finally, Figure 6 shows a correct solution. The three examples in Figures 2-7 are representative of the types of solutions submitted by all of my participants.

5 Feedback to Students

The feedback given focused not only on what students did correctly, but also on their misconceptions and instances where supporting work was unclear or incomplete. Most students were able to set up an equation that made sense based on the ratios or fractions they identified. The feedback complimented their algebraic abilities, but also made sure the students knew to go back and look at the ratios carefully. The student whose work is illustrated in Figure 2 found and checked her answer. Since the check appeared to "work," she was confident that her answer was correct. Once she got feedback that her answer was wrong, she became intrigued (and more than a little shocked). This immediately inspired her to go back, re-check her work, and consider the feedback that was given. This was similar for several other students. Another was so intrigued by the problem and curious to know how he got it wrong that he presented the problem to his calculus teacher. Remarkably, the teacher was also stumped until further guidance was provided regarding part-to-part ratios. The teacher then posed the problem to his entire calculus class.

6 In Conclusion

Common misconceptions regarding ratio and fractions make the *Points in Line Segment* task more intriguing than a routine classroom exercise. Students with a strong understanding of ratios, typically younger students or those no longer taking mathematics courses, performed well. Upper level students and those with more extensive content knowledge were initially confident about their answers and expressed genuine concern (and denial) when they were told that their work was incorrect. This inspired them to look at the problem again.

It is important to recognize that, although the task seems quite simple, it requires significant mathematical knowledge and problem solving skills. The task, while seemingly easy to understand, was challenging to interpret correctly and helped sustain student interest. It was phenomenal to see them working on math problems that they genuinely found interesting. The students' enthusiasm and effort have convinced me that rich tasks should play a more meaningful role in high school mathematics classrooms.

References

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